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On a repairable system with an unreliable service station—Bayesian approach

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ABSTRACT

System characteristics of a two-unit repairable system are studied from a Bayesian viewpoint with different types of priors assumed for unknown parameters, in which the service station is unreliable. Times to failure and times to repair of the operating units are assumed to follow exponential distributions. In addition, failure times and repair times of the service station also follow exponential distributions. When times to failure and times to repair of operating units, failure times and repair times of the service station are with uncertain parameters, a Bayesian approach is adopted to evaluate system characteristics. Monte Carlo simulation is used to derive the posterior distribution for the mean time to system failure and the steady-state availability. Some numerical experiments are performed to illustrate the results derived in this paper.

1. Introduction

The mean time to failure (or *MTTF*) and steady-state availability (or $A(\infty)$) have widely been analyzed in the literature because of their prevalence in power plants, manufacturing systems, and industrial systems. Maintaining a high or required level of reliability and/or availability is often an essential requisite. Two redundant repairable systems have been studied extensively in the past (Birolini [1], Yearout et al. [2], and detailed bibliography is found in Sztrik [3]). A number of authors have investigated two-unit redundant systems under different assumptions (see Goel and Shrivastava [4], Shi and Li [5], de Almeida and Campello de Souza [6], Gururajan and Srinivasan [7], Shi and Liu [8], Rajamanickam and Chandrasekar [9], Billinton and Pan [10], Sridharan and Mohanavadivu [11], Yadavalli et al. [12], and Seo et al. [13]). However, in many of these models the service station is reliable and available at all times. In contrast, an unreliable service station means that the service station is typically subject to unpredictable breakdowns. Gururajan and Srinivasan [7] examined a two-unit system with an unreliable service station where the lifetime of the functioning unit has a general distribution, while the standby unit has a phase-type distribution. Statistical characteristics, such as reliability function and availability function, are also provided by Gururajan and Srinivasan [7].

In the literature cited above, times to failure and times to repair of units are required to follow certain probability distribution with known parameters. However, in many real-world applications, distribution parameters are usually either unknown or uncertain. In this case, it is necessary to select an appropriate estimation method to accurately calculate the parameters of failure time distribution and repair time distribution. A great deal of study has so far focused on constructing an effective confidence interval for availability of a repairable system under the assumption of various failure time distributions

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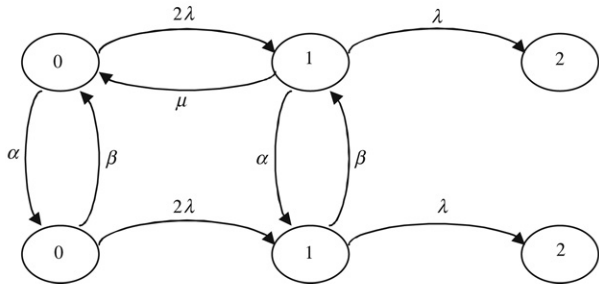

Fig. 1. The state transition diagram for the reliability model.

and repair time distributions with unknown parameters. Several confidence intervals for availability were proposed by Jie [14], Masters et al. [15], Yadavalli et al. [12] and others. Recently, Chandrasekhar et al. [16] derived a consistent asymptotically normal estimator and an asymptotic confidence interval for steady-state availability of a two-unit cold standby system in which the failure rate of the unit while online is a constant and the repair time distribution is two-stage Erlangian. In addition, some authors consider a Bayesian approach that incorporates prior knowledge for system parameters, based on past experience with similar reliability data and this prior knowledge can be mathematically translated into suitable prior density. Yadavalli et al. [17] used a Bayesian approach to study a two-unit system with common-cause shock failures by considering different prior distributions on the parameters of exponential failure and repair patterns. Their Bayesian studies focused on the steady-state availability of two different configurations (series and parallel). This paper extends their statistical inference for system availability to encompass other useful system characteristics that more accurately reflects real systems.

This paper is organized as follows. Section 2 presents a detailed model description and introduces reliability and availability characteristics of the repairable system. The Bayesian approach with different prior distributions to the system parameters is developed in Section 3. In Section 4, some numerical examples are performed to illustrate posterior analysis using Monte Carlo simulation methods and asymptotic normal results. Conclusions are drawn in Section 5.

2. Model description, reliability, and availability

We consider a repairable system with two operating units and an unreliable service station. Each operating unit fails independently of the other. Time to failure of the operating units is assumed to follow an exponential distribution with rate parameter λ. Whenever a unit fails, it immediately enters the service station, where it is served (repaired) in order of breakdown. Time to repair of a failed unit is exponentially distributed with rate parameter μ. The service station may break down at any time with breakdown rate α. Whenever the service station breaks down, it is immediately repaired with repair rate β. Breakdown times and repair times of the service station are assumed to be exponentially distributed. It is assumed that the service station can serve only one failed unit at a time and service (repair) is independent of unit failures. If the service station fails, then failed units must wait until the service station is repaired. If repair of a failed unit is interrupted by a breakdown, repair resumes as soon as the service station is available or the repair completion terminates.

In order to develop the differential equations to govern the repairable system, we first introduce some notations:

$N(t)$: number of the failed units at time t,

$\Lambda(t)$: the state of service station at time t,

where

$$\Lambda(t) = \begin{cases} 0, & \text{if the service station is available at time } t, \\ 1, & \text{if the service station is broken down at time } t. \end{cases}$$

The set $\{(\Lambda(t), N(t)); t \geq 0\}$ is a continuous-time Markovian process on the state space $\Omega = \{(\ell, n); \ell = 0, 1, n = 0, 1, 2\}$. We define

$$P_{\ell,n}(t) = \Pr[\Lambda(t) = \ell, N(t) = n],$$

which is the probability that exactly n units are failed at time t, when the service station is in the state ℓ.

2.1. The reliability function and mean time to system failure

In this subsection, we want to investigate the mean time to system failure. The state transition diagram depicted in Fig. 1. Using birth and death process, and relating the state of the system at time t and $t+\mathrm{d}t$, we have the following set of differential equations:

$$\frac{\mathrm{d}P_{0,0}(t)}{\mathrm{d}t} = -(2\lambda+\alpha)P_{0,0}(t) + \mu P_{0,1}(t) + \beta P_{1,0}(t), \tag{1}$$

$$\frac{\mathrm{d}P_{0,1}(t)}{\mathrm{d}t} = -(\lambda+\mu+\alpha)P_{0,1}(t) + 2\lambda P_{0,0}(t) + \beta P_{1,1}(t), \tag{2}$$

$$\frac{\mathrm{d}P_{0,2}(t)}{\mathrm{d}t} = \lambda P_{0,1}(t), \tag{3}$$

$$\frac{\mathrm{d}P_{1,0}(t)}{\mathrm{d}t} = -(2\lambda+\beta)P_{1,0}(t) + \alpha P_{0,0}(t), \tag{4}$$

$$\frac{\mathrm{d}P_{1,1}(t)}{\mathrm{d}t} = -(\lambda+\beta)P_{1,1}(t) + \alpha P_{0,1}(t) + 2\lambda P_{1,0}(t), \tag{5}$$

$$\frac{\mathrm{d}P_{1,2}(t)}{\mathrm{d}t} = \lambda P_{1,1}(t). \tag{6}$$

Let $\tilde{P}_{i,j}(s)$ be the Laplace transform of $P_{i,j}(t)$, $i = 0, 1, j = 0, 1, 2$. Taking the Laplace transforms on both sides of (1)–(6) and using the initial conditions, $P_{0,0}(0) = 1$, $P_{0,1}(0) = P_{0,2}(0) = P_{1,0}(0) = P_{1,1}(0) = P_{1,2}(0) = 0$. The following equations come out recursively from (4), (1), (3), (2) and (6), respectively

$$\tilde{P}_{0,0}(s) = \frac{s+2\lambda+\beta}{\alpha}\tilde{P}_{1,0}(s), \tag{7}$$

$$\tilde{P}_{0,1}(s) = \frac{(s+2\lambda)(s+2\lambda+\alpha+\beta)}{\alpha\mu}\tilde{P}_{1,0}(s) - \frac{1}{\mu}, \tag{8}$$

$$\tilde{P}_{0,2}(s) = \frac{\lambda(s+2\lambda)(s+2\lambda+\alpha+\beta)}{\alpha\mu s}\tilde{P}_{1,0}(s) - \frac{\lambda}{\mu s}, \tag{9}$$

$$\tilde{P}_{1,1}(s) = \frac{(s+2\lambda)(s+2\lambda+\alpha+\beta)+2\lambda\mu}{\mu(s+\lambda+\beta)}\tilde{P}_{1,0}(s) - \frac{\alpha}{\mu(s+\lambda+\beta)}, \tag{10}$$

$$\tilde{P}_{1,2}(s) = \frac{\lambda[(s+2\lambda)(s+2\lambda+\alpha+\beta)+2\lambda\mu]}{\mu s(s+\lambda+\beta)}\tilde{P}_{1,0}(s) - \frac{\alpha\lambda}{\mu s(s+\lambda+\beta)}, \tag{11}$$

where

$$\tilde{P}_{1,0}(s) = \frac{\alpha xy - \alpha^2\beta}{xyz(s+2\lambda) - 2\lambda\mu x(x+\lambda) - \alpha\beta(s+2\lambda)z - 2\lambda\mu\alpha\beta}, \tag{12}$$

and

$$x = s+\lambda+\beta, \qquad y = s+\lambda+\alpha+\mu, \qquad z = s+2\lambda+\alpha+\beta. \tag{13}$$

Let Z be the random variable denoting time to failure of the system; then the probability that the system fails at or before time t is $P_{0,2}(t) + P_{1,2}(t)$, that is, $Z(t) = P_{0,2}(t) + P_{1,2}(t)$. Thus, the reliability function can be expressed as

$$R(t) = 1 - Z(t) = 1 - P_{0,2}(t) - P_{1,2}(t), \quad t \geq 0. \tag{14}$$

Differentiating (14) with respect to t, we can obtain the probability density function of the failure time as follows

$$z(t) = \frac{\mathrm{d}(P_{0,2}(t) + P_{1,2}(t))}{\mathrm{d}t}. \tag{15}$$

Taking the Laplace transform of (15) and using the initial conditions, we have

$$\tilde{z}(s) = s\tilde{P}_{0,2}(s) + s\tilde{P}_{1,2}(s). \tag{16}$$

Using (9) and (11), we get

$$\tilde{z}(s) = \frac{\lambda(s+2\lambda)(s+2\lambda+\alpha+\beta)(s+\lambda+\alpha+\beta)}{\alpha\mu(s+\lambda+\beta)}\tilde{P}_{1,0} - \frac{\lambda(s+\lambda+\alpha+\beta)}{\mu(s+\lambda+\beta)}, \tag{17}$$

where $\tilde{P}_{1,0}$ is given in (12) and (13). The mean time to system failure (*MTTF*) is that the mean of the failure time t, and we have

$$MTTF = -\left.\frac{\mathrm{d}\tilde{z}(s)}{\mathrm{d}s}\right|_{s=0}.$$

Thus, the *MTTF* can be written as

$$MTTF = \frac{3\lambda\beta(\mu+2\alpha+3\lambda+\beta) + \mu\beta(\alpha+\beta) + \lambda(3\alpha+2\lambda)(\mu+3\lambda) + 3\lambda\alpha^2}{2\lambda^2(2\lambda^2+3\lambda\beta+\beta^2+3\lambda\alpha+\alpha^2+\alpha\mu+2\alpha\beta)}. \tag{18}$$

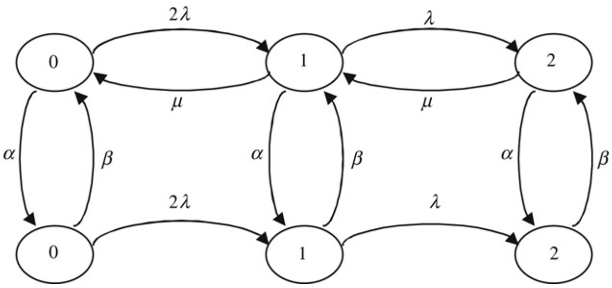

Fig. 2. The state transition diagram for the availability model.

2.2. Steady-state availability

Availability is an interesting measure to the repairable system. The state transition diagram is depicted in Fig. 2. By the same argument of Section 2.1, it finally yields

$$\frac{\mathrm{d}P_{0,0}(t)}{\mathrm{d}t} = -(2\lambda+\alpha)P_{0,0}(t) + \mu P_{0,1}(t) + \beta P_{1,0}(t), \tag{19}$$

$$\frac{\mathrm{d}P_{0,1}(t)}{\mathrm{d}t} = -(\lambda+\mu+\alpha)P_{0,1}(t) + 2\lambda P_{0,0}(t) + \mu P_{0,2}(t) + \beta P_{1,1}(t), \tag{20}$$

$$\frac{\mathrm{d}P_{0,2}(t)}{\mathrm{d}t} = -(\mu+\alpha)P_{0,2}(t) + \lambda P_{0,1}(t) + \beta P_{1,2}(t), \tag{21}$$

$$\frac{\mathrm{d}P_{1,0}(t)}{\mathrm{d}t} = -(2\lambda+\beta)P_{1,0}(t) + \alpha P_{0,0}(t), \tag{22}$$

$$\frac{\mathrm{d}P_{1,1}(t)}{\mathrm{d}t} = -(\lambda+\beta)P_{1,1}(t) + \alpha P_{0,1}(t) + 2\lambda P_{1,0}(t), \tag{23}$$

$$\frac{\mathrm{d}P_{1,2}(t)}{\mathrm{d}t} = -\beta P_{1,2}(t) + \alpha P_{0,2}(t) + \lambda P_{1,1}(t). \tag{24}$$

In steady-state, we define

$$P_{0,n} = \lim_{t\to\infty} P_{0,n}(t) \quad \text{and} \quad P_{1,n} = \lim_{t\to\infty} P_{1,n}(t), \quad n = 0, 1, 2. \tag{25}$$

In steady-state, $P_{0,n}(t)$ and $P_{1,n}(t)$ are independent of t, hence we have

$$\frac{\mathrm{d}P_{i,j}(t)}{\mathrm{d}t} = 0, \quad \text{where } i = 0, 1;\ j = 0, 1, 2. \tag{26}$$

Combining (25) and (26) with (22), (19), (23), (20) and (21), respectively, we recurively obtain $P_{1,0}$, $P_{0,1}$, $P_{1,1}$, $P_{0,2}$, and $P_{1,2}$ in terms of $P_{0,0}$, after some arduous algebraic rearrangements,

$$P_{1,0} = \frac{\alpha}{2\lambda+\beta}P_{0,0},$$

$$P_{0,1} = \frac{2\lambda(2\lambda+\alpha+\beta)}{\mu(2\lambda+\beta)}P_{0,0},$$

$$P_{1,1} = \frac{2\alpha\lambda(2\lambda+\alpha+\beta+\mu)}{\mu(2\lambda+\beta)(\lambda+\beta)}P_{0,0},$$

$$P_{0,2} = \frac{2\lambda^2[(2\lambda+\alpha+\beta)(\lambda+\alpha+\beta)+\alpha\mu]}{\mu^2(2\lambda+\beta)(\lambda+\beta)}P_{0,0},$$

$$P_{1,2} = \frac{2\alpha\lambda^2[(2\lambda+\alpha+\beta)(\lambda+\alpha+\beta+\mu)+\mu(\alpha+\mu)]}{\mu^2\beta(2\lambda+\beta)(\lambda+\beta)}P_{0,0}.$$

By the normalizing condition, we have

$$P_{0,0} = \frac{\mu^2\beta(\lambda+\beta)(2\lambda+\beta)}{K(\alpha+\beta)}.$$

Thus, the availability is given by

$$\begin{aligned} A &= P_{0,0} + P_{0,1} + P_{1,0} + P_{1,1} \\ &= \frac{\mu\beta\{(\alpha+\beta)[\lambda(6\lambda+2\beta+3\mu+2\alpha)+\mu\beta]+\lambda^2(4\lambda+2\mu)\}}{K(\alpha+\beta)}, \end{aligned} \tag{27}$$

where

$$K = (2\alpha^2\lambda^2 + 2\mu\beta\alpha\lambda + 4\mu\lambda^2\alpha + 4\beta\lambda^2\alpha + 6\lambda^3\alpha + \mu^2\beta^2 + 2\lambda^2\beta^2 + 3\mu^2\beta\lambda + 4\lambda^3\mu + 2\mu\beta^2\lambda + 6\mu\beta\lambda^2 + 2\mu^2\lambda^2 + 4\lambda^4 + 6\lambda^3\beta).$$

3. Bayesian approach of *MTTF* and steady-state availability

In this section, we propose a Bayesian approach for the case that the parameters λ, μ, α, and β, of the repairable system are unknown and have to be estimated from appropriate prior distributions. First, we establish the likelihood function of λ, μ, α, and β with no prior information.

3.1. Likelihood function

The time to failure and repair of operating units, and the breakdown time and repair time of the service station are independently distributed random variables. Let $\tilde{U}_1 = (U_{11}, U_{12}, \ldots, U_{1,n_1})$ and $\tilde{U}_2 = (U_{21}, U_{22}, \ldots, U_{2,n_2})$ be the random samples of sizes n_1, n_2 respectively for failure times and repair times of operating units. Let $\tilde{U}_3 = (U_{31}, U_{32}, \ldots, U_{3,n_3})$ and $\tilde{U}_4 = (U_{41}, U_{42}, \ldots, U_{4,n_4})$ be the random samples of sizes n_3, n_4 respectively for breakdown times and repair times of the service station. All samples are drawn from exponential populations. The likelihood function of λ, μ, α and β can be obtained using the following formulae:

$$f(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \tilde{U}_4 \mid \lambda, \mu, \alpha, \beta) \propto \lambda^{n_1}\mu^{n_2}\alpha^{n_3}\beta^{n_4}\mathrm{e}^{-(\lambda T_1+\mu T_2+\alpha T_3+\beta T_4)}, \tag{28}$$

where (T_1, T_2, T_3, T_4) is sufficient for $(\lambda, \mu, \alpha, \beta)$, and $T_i = \sum_{j=1}^{n_i} U_{ij}$, $i = 1, 2, 3, 4$.

3.2. Two-parameter gamma prior

In Bayesian statistics, a prior distribution is multiplied by a likelihood function and then produces a posterior distribution. A conjugate prior is one which produces a posterior distribution which is of the same type as the prior. Conjugate priors are often very flexible and convenient. Prior distribution with two parameters ν and γ is the conjugate prior for the exponential model. If we have real data from previous testing done on this system, this is the prior knowledge. Simply set the parameters ν and $1/\gamma$ equal to the total number of failures and the total time, respectively, from all the previous data. Therefore, the gamma distribution is an appropriate prior distribution.

An appropriate conjugate prior distribution for λ is a gamma distribution $G(\nu_1, \gamma_1)$, which is given by

$$p(\lambda) = \frac{\gamma_1^{\nu_1}\lambda^{\nu_1-1}\mathrm{e}^{-\gamma_1\lambda}}{\Gamma(\nu_1)}, \quad \text{for } \lambda > 0, \tag{29}$$

where $\nu_1 > 0, \gamma_1 > 0$ are special parameters, $\mathrm{E}(\lambda) = \nu_1/\gamma_1$ and $\mathrm{Var}(\lambda) = \nu_1/\gamma_1^2$. According to Bayesian theory and using (28) and (29), the posterior distribution of λ given T_1 is given by

$$h(\lambda \mid T_1) = \frac{(T_1+\gamma_1)^{n_1+\nu_1}\lambda^{n_1+\nu_1-1}\mathrm{e}^{-\lambda(T_1+\gamma_1)}}{\Gamma(n_1+\nu_1)}, \tag{30}$$

which is the density of a gamma distribution with parameters $n_1 + \nu_1$ and $T_1 + \gamma_1$, $\lambda \mid T_1 \sim G(n_1 + \nu_1, T_1 + \gamma_1)$.

A natural estimator for λ is the mean of the posterior distribution, which we denoted by $\hat{\lambda}_B = (n_1+\nu_1)/(T_1+\gamma_1)$. The prior distribution has a mean ν_1/γ_1, which would be the estimate of λ before observing the data. Ignoring the prior information, we would probably use n_1/T_1 as the estimate of λ. The posterior estimate of λ combines all of this information. We can represent $\hat{\lambda}_B$ as a linear combination of the prior mean and n_1/T_1 with weights $\gamma_1/(T_1 + \gamma_1)$ and $T_1/(T_1 + \gamma_1)$, respectively. Thus, we observe that the weight of the prior mean will decrease as n_1 increases, that is, the effect of hyperparameters of the prior distribution for the posterior mean will get smaller if n_1 is large.

Similarly, $G(\nu_i, \gamma_i)$, $i = 2, 3, 4$, are assumed as prior distribution for μ, α, and β, respectively. We assume that prior distributions of all system parameters are independent. Thus, the joint distribution of λ, μ, α, and β is taken to be the product of prior distributions of each parameter. Proceeding above derivations listed, we obtain the joint posterior distribution given by

$$\lambda, \mu, \alpha, \beta \mid T_1, T_2, T_3, T_4 \sim G(n_1 + \nu_1, T_1 + \gamma_1) \cdot G(n_2 + \nu_2, T_2 + \gamma_2) \cdot G(n_3 + \nu_3, T_3 + \gamma_3) \cdot G(n_4 + \nu_4, T_4 + \gamma_4). \tag{31}$$

3.3. Standard gamma prior

If we use standard gamma density $G(\nu_i, 1)(i = 1, 2, 3, 4)$ as the prior distribution of λ, μ, α, and β, respectively, then the joint posterior distribution in (31) becomes

$$\lambda, \mu, \alpha, \beta \mid T_1, T_2, T_3, T_4 \sim G(n_1 + \nu_1, T_1 + 1) \cdot G(n_2 + \nu_2, T_2 + 1) \cdot G(n_3 + \nu_3, T_3 + 1) \cdot G(n_4 + \nu_4, T_4 + 1). \tag{32}$$

3.4. Jeffreys' prior

If the noninformative prior distribution of λ, μ, α and β is specified as follows

$$p(\lambda, \mu, \alpha, \beta) \propto \frac{1}{\lambda\mu\alpha\beta}, \quad \lambda, \mu, \alpha, \beta > 0,$$

then the joint posterior distribution is

$$h(\lambda, \mu, \alpha, \beta \mid T_1, T_2, T_3, T_4) \propto \lambda^{n_1-1}\mu^{n_2-1}\alpha^{n_3-1}\beta^{n_4-1}\mathrm{e}^{-(\lambda T_1+\mu T_2+\alpha T_3+\beta T_4)}. \tag{33}$$

3.5. Beta distribution of second kind prior

Another appropriate prior distribution for λ is a beta distribution of the second kind (beta-prime or inverted-beta-2; denoted by $BP(m_1, r_1)$) which is given by

$$g(\lambda) = \frac{\lambda^{m_1-1}}{B(m_1, r_1)(1+\lambda)^{m_1+r_1}}, \quad \text{for } \lambda > 0, m_1 > 0, r_1 > 0.$$

Suitable values of the hyperparameters m_1, r_1 are

$$m_1 = \frac{\omega_1(\omega_1 + \omega_1^2 + \sigma_1^2)}{\sigma_1^2}, \qquad r_1 = \frac{\omega_1 + \omega_1^2 + 2\sigma_1^2}{\sigma_1^2},$$

where ω_1 and σ_1^2 are the prior mean and prior variance, respectively. According to Bayesian theory and using (28) with the beta distribution of the second kind prior, the posterior distribution of λ is obtained as follows

$$h(\lambda \mid T_1) = \frac{\lambda^{n_1+m_1-1}\mathrm{e}^{-\lambda T_1}}{\Gamma(n_1+m_1)U(n_1+m_1, n_1-r_1+1, T_1)(1+\lambda)^{m_1+r_1}},$$

which is a generalized gamma distribution as defined by Agarwal and Kalla [18], and where $U(a, b, z)$ is the confluent hypergeometric function given by

$$U(a, b, z) = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}(1+x)^{b-a-1}\mathrm{e}^{-zx}\mathrm{d}x.$$

Similarly, $BP(m_i, r_i)$, $i = 2, 3, 4$, are assumed as prior distribution for μ, α and β, respectively. Prior distributions of all system parameters are independent. Thus, the joint distribution of λ, μ, α and β is taken to be the product of the prior distributions of each parameter. Proceeding analogously, we obtain the joint posterior distribution which is given by

$$\lambda, \mu, \alpha, \beta \mid T_1, T_2, T_3, T_4 \propto \frac{\lambda^{n_1+m_1-1}\mu^{n_2+m_2-1}\alpha^{n_3+m_3-1}\beta^{n_4+m_4-1}\mathrm{e}^{-(\lambda T_1+\mu T_2+\alpha T_3+\beta T_4)}}{(1+\lambda)^{m_1+r_1}(1+\mu)^{m_2+r_2}(1+\alpha)^{m_3+r_3}(1+\beta)^{m_4+r_4}}. \tag{34}$$

From each of the joint posterior distributions (31)–(34), λ, μ, α and β can be generated by means of the Monte Carlo simulation method. The *MTTF* and $A(\infty)$ are obtained by substituting $(\lambda, \mu, \alpha, \beta)$ into (18) and (27), respectively. If M pairs $(MTTF, A(\infty))$ are drawn, posterior distributions of *MTTF* and $A(\infty)$ for the repairable system can be obtained, from which the posterior mean (PM) and the highest posterior density (HPD) intervals can be calculated. Finally, we make some numerical comparisons between the posterior and asymptotic performances of *MTTF* and $A(\infty)$.

4. Simulation study and comparisons

In this section we use simulation results to discuss the posterior performances of *MTTF* and $A(\infty)$ for the repairable system with an unreliable service station. We set $n_1 = n_2 = n_3 = n_4 = n$. We run 10 000 simulations for each prior distribution considered in Section 3. For each simulation run, we first generate values from the assumed prior distributions. These simulated values are then used as parameter values for the time between failures, the repair time, and the breakdown time distributions. A sample of size n is then generated from each of the four time variables. The PM and HPD intervals are then computed. The tables list the the mean of these 10 000 HPD PM, its estimated standard deviation $s/\sqrt{10\,000}$ and the proportion of the 10 000 HPD that covered the simulated parameter value. The samples were generated using appropriate subroutines of S-PLUS 6.1.

Tables 1 and 2 give the PM and the HPD intervals of *MTTF* and $A(\infty)$, respectively, for the various values of the parameters λ, μ, α and β when the Jeffreys' prior is assumed and $n = 30$. From these two tables, it is clear that a large λ or α or a small μ or β induces a smaller PM and the failure rate λ has a better effect on the mean time to failure or the steady-state availability. Compared with the true values, the 95% HPD intervals cover the true values. Tables 3 and 4 give the PM and the HPD intervals of *MTTF* and $A(\infty)$, respectively, for various sample sizes and $\lambda = 0.1$, $\mu = 2$, $\alpha = 0.05$, $\beta = 8$ when the two-parameter gamma prior with various hyperparameters is assumed. From these two tables, it is evident that the larger the sample size, the narrower the HPD intervals and the PMs are closer to the true values 114.23 and 0.99541 of *MTTF* and

Table 1
Posterior mean (PM) and HPD interval for *MTTF* ($n = 30$, and Jeffreys' prior is assumed)

λ	μ	α	β	True value	PM	St.dev.	95% HPD	
0.01	2	0.01	5	10 122.14	11 892.27	7 348.14	2093.12	25 665.13
0.01	4	0.01	5	20 078.51	23 606.87	15 082.25	4217.01	52 395.67
0.01	2	0.05	5	10 012.55	11 651.19	7 264.66	2129.93	25 790.10
0.01	2	0.1	5	9 879.55	11 613.81	7 122.25	2228.03	25 803.48
0.01	2	0.01	8	10 134.41	12 072.81	7 704.26	1926.53	26 729.77
0.05	2	0.01	5	428.89	502.54	308.28	85.89	1 092.10
0.05	4	0.01	5	827.17	969.69	597.81	145.04	2 107.81
0.05	2	0.05	5	424.54	497.54	304.451	92.84	1 085.06
0.05	2	0.1	5	419.25	486.34	294.61	98.49	1 060.62
0.05	2	0.01	8	429.38	505.69	314.96	97.29	1 128.20
0.1	2	0.01	5	114.73	132.06	78.89	25.82	282.75
0.1	4	0.01	5	214.30	251.54	155.63	47.92	549.08
0.1	2	0.05	5	113.65	131.83	77.47	30.43	284.99
0.1	2	0.1	5	112.33	129.63	75.75	31.01	275.66
0.1	2	0.01	8	114.85	132.94	78.26	28.70	285.05

Table 2
Posterior mean (PM) and HPD interval for $A(\infty)$ ($n = 30$, and Jeffreys' prior is assumed)

λ	μ	α	β	True value	PM	St.dev.	95% HPD	
0.01	2	0.01	5	0.99995	0.99994	0.00006	0.99988	0.99999
0.01	4	0.01	5	0.99999	0.99998	0.00001	0.99996	1.00000
0.01	2	0.05	5	0.99995	0.99993	0.00006	0.99983	1.00000
0.01	2	0.1	5	0.99995	0.99993	0.00006	0.99982	1.00000
0.01	2	0.01	8	0.99995	0.99994	0.00006	0.99983	1.00000
0.05	2	0.01	5	0.99880	0.99848	0.00122	0.99614	0.99986
0.05	4	0.01	5	0.99969	0.99960	0.00032	0.99898	0.99997
0.05	2	0.05	5	0.99878	0.99842	0.00126	0.99598	0.99986
0.05	2	0.1	5	0.99874	0.99839	0.00129	0.99607	0.99987
0.05	2	0.01	8	0.99881	0.99848	0.00121	0.99617	0.99987
0.1	2	0.01	5	0.99545	0.99428	0.00450	0.98587	0.99958
0.1	4	0.01	5	0.99880	0.99846	0.00124	0.99615	0.99987
0.1	2	0.05	5	0.99535	0.99409	0.00446	0.98548	0.99940
0.1	2	0.1	5	0.99522	0.99392	0.00466	0.98503	0.99937
0.1	2	0.01	8	0.99546	0.99433	0.00435	0.98604	0.99942

Table 3
Posterior mean (PM) of *MTTF* for $\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$

n	Two-parameter gamma prior								
	$(\nu_2, \gamma_2) = (20, 10)$; $(\nu_3, \gamma_3) = (0.5, 10)$; $(\nu_4, \gamma_4) = (80, 10)$								
	$(\nu_1, \gamma_1) = (0.5, 5)$			$(\nu_1, \gamma_1) = (2, 20)$			$(\nu_1, \gamma_1) = (14.3, 143)$		
	PM	95% HPD		PM	95% HPD		PM	95% HPD	
10	167.10	7.54	476.75	156.95	12.92	426.65	130.94	35.31	226.04
30	130.11	29.56	269.99	129.02	32.37	258.19	124.36	43.85	227.86
50	124.06	45.13	229.11	112.84	40.46	220.25	121.82	53.18	209.65
100	119.37	60.93	187.53	119.37	58.86	186.89	118.22	65.62	186.44
500	115.11	87.54	145.33	115.43	88.68	146.03	115.09	88.11	144.31
1000	114.67	94.95	138.81	114.55	94.02	135.07	114.70	95.21	136.10

(True value of *MTTF* = 114.23).

Table 4
Posterior mean (PM) of $A(\infty)$ for $\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$

n	Two-parameter gamma prior								
	$(\nu_2, \gamma_2) = (20, 10)$; $(\nu_3, \gamma_3) = (0.5, 10)$; $(\nu_4, \gamma_4) = (80, 10)$								
	$(\nu_1, \gamma_1) = (0.5, 5)$			$(\nu_1, \gamma_1) = (2, 20)$			$(\nu_1, \gamma_1) = (14.3, 143)$		
	PM	95% HPD		PM	95% HPD		PM	95% HPD	
10	0.99311	0.97945	0.99984	0.99350	0.98216	0.99980	0.99455	0.98792	0.99927
30	0.99453	0.98751	0.99905	0.99458	0.98790	0.99901	0.99480	0.98954	0.99890
50	0.99482	0.98964	0.99878	0.99485	0.98981	0.99872	0.99492	0.99039	0.99848
100	0.99508	0.99168	0.99812	0.99511	0.99152	0.99808	0.99512	0.99188	0.99808
500	0.99535	0.99380	0.99684	0.99535	0.99380	0.99681	0.99534	0.99371	0.99676
1000	0.99537	0.99427	0.99642	0.99537	0.99426	0.99639	0.99537	0.99428	0.99640

(True value of $A(\infty) = 0.99541$).

Table 5
Posterior mean (PM) and HPD interval for *MTTF* ($\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$; and Jeffreys' prior is assumed)

n	PM	St.dev.	99%HPD		95%HPD	
10	181.92	242.72	6.00	1183.73	7.05	554.07
30	133.42	77.88	21.11	402.45	31.98	288.89
50	125.03	54.74	35.61	313.11	42.60	234.07
100	119.01	35.13	49.32	224.68	59.86	190.44
500	115.24	15.04	80.00	158.11	86.20	144.23
1000	114.51	10.55	89.23	144.21	94.16	134.90

Table 6
Posterior mean (PM) and HPD interval for $A(\infty)$ ($\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$; and Jeffreys' prior is assumed)

n	PM	St.dev.	99%HPD		95%HPD	
10	0.99092	0.01422	0.93042	0.99999	0.96765	0.99996
30	0.99418	0.00452	0.97703	0.99970	0.98560	0.99936
50	0.99467	0.00310	0.98422	0.99917	0.98864	0.99889
100	0.99507	0.00197	0.98862	0.99853	0.99120	0.99827
500	0.99534	0.00080	0.99313	0.99717	0.99373	0.99676
1000	0.99538	0.00056	0.99383	0.99665	0.99424	0.99642

Table 7
Posterior mean (PM) of *MTTF* for $\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$

n	Beta distribution of the second kind prior								
	$(m_2, r_2) = (22.04, 12.02)$; $(m_3, r_3) = (0.18125, 4.625)$; $(m_4, r_4) = (296, 38)$								
	$(m_1, r_1) = (0.32, 4.2)$			$(m_1, r_1) = (1.2, 13)$			$(m_1, r_1) = (11.1, 112)$		
	PM	95%HPD		PM	95%HPD		PM	95%HPD	
10	516.30	10.56	1554.32	433.53	13.40	1309.11	111.21	4.87	309.11
30	214.15	43.25	461.27	202.33	40.05	426.43	117.32	30.46	242.44
50	171.70	56.24	324.58	166.57	55.20	312.85	118.56	41.61	213.68
100	142.84	70.40	227.96	139.96	67.11	224.99	116.89	57.10	183.20
500	120.14	90.99	151.86	119.24	90.08	150.56	115.17	85.80	143.49
1000	117.05	96.93	138.38	116.73	95.97	138.21	114.71	93.99	135.31

(True value of *MTTF* $= 114.23$).

Table 8
Posterior mean (PM) of $A(\infty)$ for $\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$

n	Beta distribution of the second kind prior								
	$(m_2, r_2) = (22.04, 12.02)$; $(m_3, r_3) = (0.18125, 4.625)$; $(m_4, r_4) = (296, 38)$								
	$(m_1, r_1) = (0.32, 4.2)$			$(m_1, r_1) = (1.2, 13)$			$(m_1, r_1) = (11.1, 112)$		
	PM	95%HPD		PM	95%HPD		PM	95%HPD	
10	0.99910	0.99696	0.99999	0.99897	0.99654	0.99999	0.99670	0.98939	0.99996
30	0.99803	0.99521	0.99980	0.99791	0.99501	0.99979	0.99653	0.99191	0.99953
50	0.99743	0.99468	0.99945	0.99731	0.99453	0.99946	0.99622	0.99227	0.99925
100	0.99668	0.99429	0.99877	0.99661	0.99404	0.99882	0.99593	0.99285	0.99844
500	0.99574	0.99433	0.99709	0.99572	0.99428	0.99707	0.99555	0.99401	0.99690
1000	0.99557	0.99449	0.99659	0.99557	0.99452	0.99658	0.99549	0.99439	0.99648

(True value of $A(\infty) = 0.99541$).

$A(\infty)$, respectively, as the sample size increases. We found that the PM are more stable and accurate when the sample size is large. We also found that the PM are close to the true values even though the sample size is small when we use the two-parameter gamma distribution with the hyperparameters $(\nu_1, \gamma_1) = (14.3, 143)$, $(\nu_2, \gamma_2) = (20, 10)$, $(\nu_3, \gamma_3) = (0.5, 10)$, and $(\nu_4, \gamma_4) = (80, 10)$ as prior.

In Table 5, we give the PM and the HPD intervals of *MTTF* for various sample sizes when the Jeffreys' prior is assumed. From this table, it is evident the HPD intervals of *MTTF* are much narrower as the sample size increases. Similar results for $A(\infty)$ are found in Table 6. In Tables 7 and 8, we use the beta distribution of the second kind prior with various values of the hyperparameters and sample sizes to get Bayesian estimates for *MTTF* and $A(\infty)$, respectively, when $\lambda = 0.1$, $\mu = 2$, $\alpha = 0.05$, $\beta = 8$. The results of these two tables are similar to the results of Tables 5 and 6. In Tables 3–8, we found that when the sample size is large, the PM is close to the true value when Jeffrey' prior is assumed or no matter what values the hyperparameters took when we use the the two-parameter gamma distribution and the beta distribution of the second kind as prior.

Table 9
Posterior mean (PM), HPD intervals and asymptotic confidence intervals of *MTTF* ($\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$; and standard gamma prior is assumed for the hyperparameter. True value of *MTTF* = 114.23)

n	PM	St.dev.	95%HPD		$\hat{MTTF}$	95%ACI	
10	177.35	210.81	7.03	523.70	72.28	15.13	345.24
30	131.62	76.15	26.62	278.82	175.21	78.51	390.01
50	124.32	53.57	42.58	231.98	210.03	113.59	388.37
100	119.30	35.33	58.31	191.95	86.33	58.77	126.81
500	115.28	15.12	87.86	146.03	112.80	95.34	133.45
1000	114.68	10.60	95.03	136.42	114.31	101.70	128.49

Table 10
Posterior mean (PM), HPD intervals and asymptotic confidence intervals of $A(\infty)$ ($\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$; and standard gamma prior is assumed for the hyperparameter. True value of $A(\infty) = 0.99541$)

n	PM	St.dev.	95%HPD		$\hat{A}(\infty)$	95%ACI	
10	0.99142	0.01245	0.97132	0.99994	0.99740	0.98675	0.99950
30	0.99419	0.00439	0.98560	0.99931	0.99600	0.97829	0.99926
50	0.99475	0.00298	0.98895	0.99897	0.99503	0.98583	0.99827
100	0.99507	0.00190	0.99126	0.99811	0.99322	0.98747	0.99634
500	0.99535	0.00079	0.99377	0.99677	0.99547	0.99416	0.99648
1000	0.99538	0.00056	0.99422	0.99640	0.99692	0.99631	0.99742

Table 11
The rates of coverage for HPD intervals and asymptotic confidence intervals of $A(\infty)$ and *MTTF* ($\lambda = 0.1$; $\mu = 2$; $\alpha = 0.05$; $\beta = 8$; standard gamma prior is assumed for the hyperparameter)

n	*MTTF*		$A(\infty)$	
	HPD	ACI	HPD	ACI
10	0.9513	0.8644	0.9572	0.8423
30	0.9426	0.9154	0.9501	0.8991
50	0.9437	0.9224	0.9444	0.9186
100	0.9419	0.9364	0.9452	0.9287
500	0.9447	0.9466	0.9418	0.9446
1000	0.9424	0.9491	0.9431	0.9491

Next, we compare the performances of the PM and the HPD intervals with the asymptotic estimate and the asymptotic confidence intervals (ACI) when the standard gamma prior is applied. Note that standard gamma priors are special cases of the two-parameter gamma prior with $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$. The asymptotic estimate $\hat{MTTF}$ is defined as

$$\hat{MTTF} = \frac{3\bar{U}_1\bar{U}_4(\bar{U}_2 + 2\bar{U}_3 + 3\bar{U}_1 + \bar{U}_4) + \bar{U}_2\bar{U}_4(\bar{U}_3 + \bar{U}_4)}{2\bar{U}_1^2(2\bar{U}_1^2 + 3\bar{U}_1\bar{U}_4 + \bar{U}_4^2 + 3\bar{U}_1\bar{U}_3 + \bar{U}_3^2 + \bar{U}_3\bar{U}_2 + 2\bar{U}_3\bar{U}_4)} + \frac{\bar{U}_1(3\bar{U}_3 + 2\bar{U}_1)(\bar{U}_2 + 3\bar{U}_1) + 3\bar{U}_1\bar{U}_3^2}{2\bar{U}_1^2(2\bar{U}_1^2 + 3\bar{U}_1\bar{U}_4 + \bar{U}_4^2 + 3\bar{U}_1\bar{U}_3 + \bar{U}_3^2 + \bar{U}_3\bar{U}_2 + 2\bar{U}_3\bar{U}_4)},$$

where

$$\bar{U}_1 = \hat{\lambda}, \qquad \bar{U}_2 = \hat{\mu}, \qquad \bar{U}_3 = \hat{\alpha}, \qquad \bar{U}_4 = \hat{\beta} \quad \text{and} \quad \bar{U}_i = \frac{n}{\sum_{j=1}^{n} U_{ij}}, \quad i = 1, 2, 3, 4.$$

In addition, the ACI are given by

$$\hat{MTTF} \pm z_{\alpha/2}\frac{\sigma(\hat{\Theta})}{\sqrt{n}},$$

where $\sigma^2(\hat{\Theta})$ is a consistent estimator of $\sigma^2(\Theta) = \sum_{i=1}^{4}\left[\frac{\partial MTTF}{\partial\theta_i}\right]^2\theta_i^2$ with

$$\Theta = (\theta_1, \theta_2, \theta_3, \theta_4) = \left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{\alpha}, \frac{1}{\beta}\right).$$

Since *MTTF* is constrained to be positive, we take the log transformation (see Meeker and Escobar [19]) and use the delta method to find the appropriate asymptotic estimator and the reliable ACI. The new asymptotic estimator and the new ACI are $\hat{MTTF}$ and $(\hat{MTTF} \cdot \exp(-z_{\alpha/2}\sigma(\hat{\Theta})/\sqrt{n}), \hat{MTTF} \cdot \exp(-z_{\alpha/2}\sigma(\hat{\Theta})/\sqrt{n}))$, respectively. Likewise, the asymptotic estimator and the ACI for $A(\infty)$ based on logit transformation (see Meeker and Escobar [19]) can be obtained since $0 \le A(\infty) \le 1$. Table 9 lists the PM, the HPD intervals, and the ACI of *MTTF* for various sample sizes when the standard gamma prior with

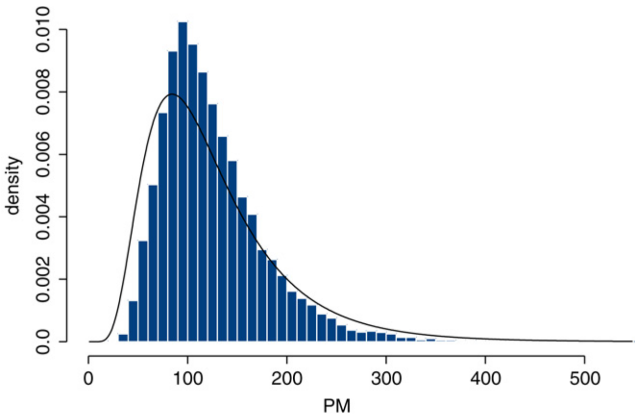

Fig. 3. Posterior distribution for *MTTF*: two-parameter gamma prior ($n = 30$).

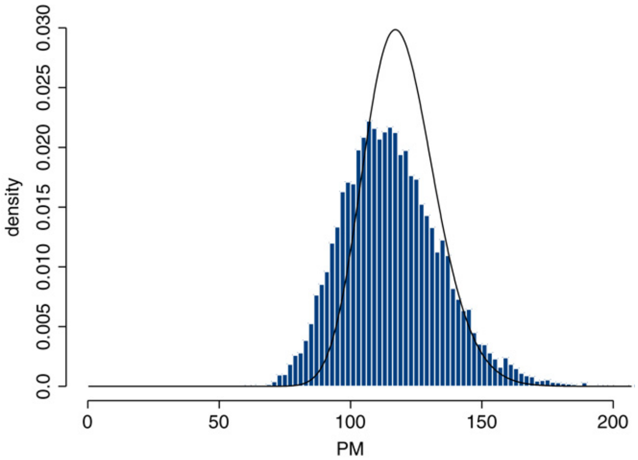

Fig. 4. Posterior distribution for *MTTF*: two-parameter gamma prior ($n = 300$).

the hyperparameters $\nu_1 = 0.1$, $\nu_2 = 2$, $\nu_3 = 0.05$, and $\nu_4 = 8$ and $\lambda = 0.1$, $\mu = 2$, $\alpha = 0.05$ and $\beta = 8$. Compared with the true value 114.23, the results shows that when the sample size is large, the PM and the asymptotic estimate $\hat{MTTF}$ are closer to the true value. When the sample size is small the HPD intervals are wider than the ACI. As expected the spreads of posterior distributions and asymptotic distributions will get smaller with increasing sample size. Similar inferences for $A(\infty)$ are obtained in Table 10.

In order to compare the reliability of the intervals for *MTTF* and $A(\infty)$ obtained by the two different methods mentioned in Tables 9 and 10, the coverage probabilities for HPD intervals and ACI are given in Table 11. Results from 10 000 replications were used in construction of the table. The coverage probability can be estimated by the proportion of the number of times the true value is contained, to the number of simulations. From this table, we observe that the coverage probability of the ACI covering the true value is smaller than 0.95 when the sample size is smaller than 50. The corresponding HPD intervals are much better, because the percentage is closer to 0.95 for the 12 situations. It is noted that the coverage probability by the asymptotic method will be closer to 0.95 as the sample size increases. This explains the phenomenon in which the ACIs for *MTTF* and $A(\infty)$ are smaller than the HPD intervals. This also indicates that the asymptotic test based on the likelihood ratio criterion will be biased toward rejecting the null hypothesis.

Lastly, we give some figures of posterior distributions to illustrate the performances of *MTTF* and $A(\infty)$. The histogram is plotted from the 10 000 PM and the curve for *MTTF* is sketched by the lognormal density function with location parameter $\log \hat{MTTF}$ and scale parameter $\sigma(\hat{\Theta}^2)/(n \cdot \hat{MTTF}^2)$. Likewise, the curve for $A(\infty)$ is sketched by the logistic density function. For convenience, Figs. 3–18 are based on $\lambda = 0.1$, $\mu = 2$, $\alpha = 0.05$, and $\beta = 8$. Figs. 3 and 4 show the posterior distribution for *MTTF* if $n = 30$ and $n = 300$, respectively, assuming the two-parameter gamma prior with the hyperparameters $(\nu_1, \gamma_1) = (14.3, 143)$, $(\nu_2, \gamma_2) = (20, 10)$, $(\nu_3, \gamma_3) = (0.5, 10)$, and $(\nu_4, \gamma_4) = (80, 10)$. Similarly, the posterior distribution for $A(\infty)$ if $n = 30$ and $n = 300$ are shown in Figs. 5 and 6, respectively. The posterior distributions for *MTTF* and $A(\infty)$ with the standard gamma with the hyperparameters $\nu_1 = 0.1$, $\nu_2 = 2$, $\nu_3 = 0.05$, and $\nu_4 = 8$ as prior are plotted in Figs. 7–10. Figs. 11–14 show the posterior distribution for *MTTF* and $A(\infty)$ if $n = 30$ and $n = 300$, respectively when Jeffreys' prior is used as prior. Figs. 15–18 show the posterior distributions when the beta distribution of the second kind with the

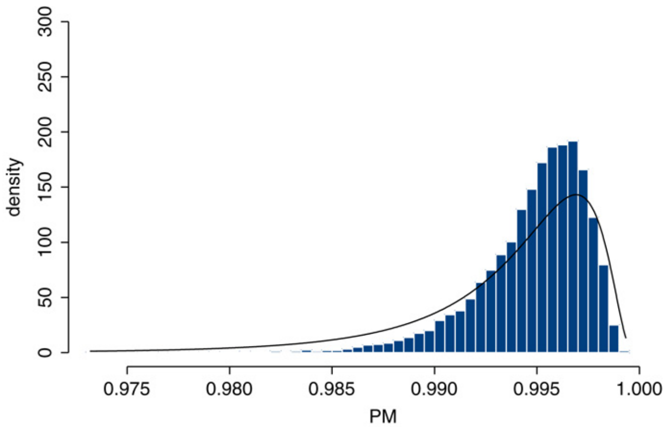

Fig. 5. Posterior distribution for $A(\infty)$: two-parameter gamma prior ($n = 30$).

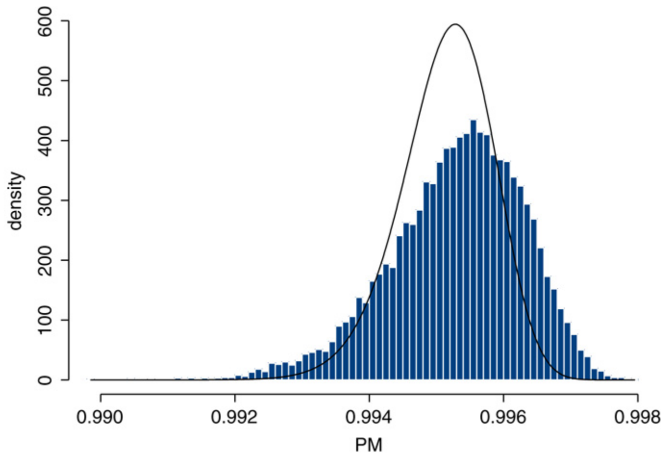

Fig. 6. Posterior distribution for $A(\infty)$: two-parameter gamma prior ($n = 300$).

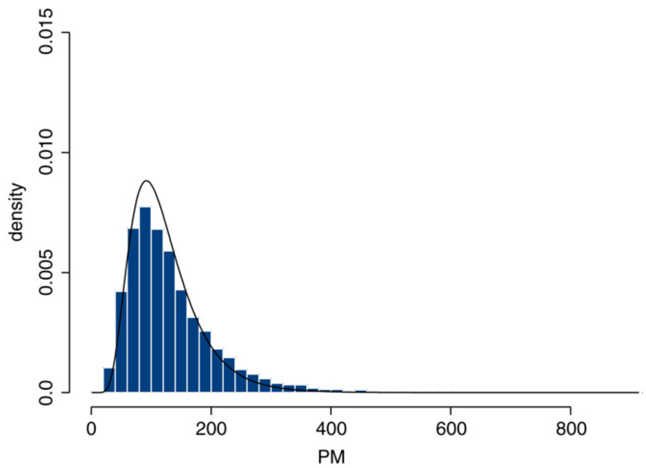

Fig. 7. Posterior distribution for *MTTF*: standard gamma prior ($n = 30$).

hyperparameters $(m_1, r_1) = (0.32, 4.2)$, $(m_2, r_2) = (22.04, 12.02)$, $(m_3, r_3) = (0.18125, 4.625)$ and $(m_4, r_4) = (296, 38)$ is used as prior. No matter what prior is chosen, the spreads of the posterior distributions for *MTTF* and $A(\infty)$ will get smaller as the sample size gets large and the histograms for the posterior distributions of *MTTF* are slightly skew to the right and the histograms of $A(\infty)$ are skew to the left.

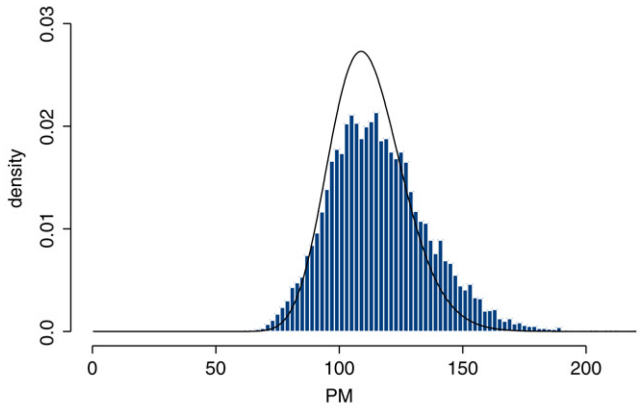

Fig. 8. Posterior distribution for *MTTF*: standard gamma prior ($n = 300$).

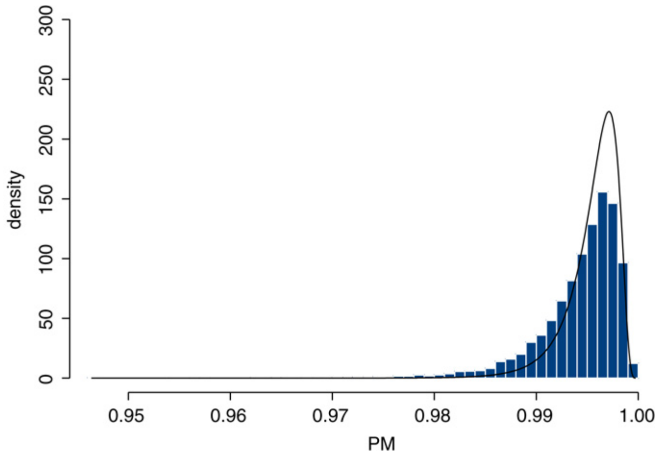

Fig. 9. Posterior distribution for $A(\infty)$: standard gamma prior ($n = 30$).

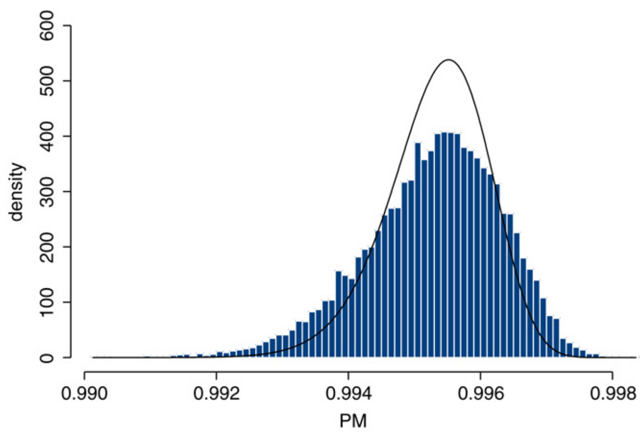

Fig. 10. Posterior distribution for $A(\infty)$: standard gamma prior ($n = 300$).

As one would expect, this indicates that the Bayesian results developed in this paper are reasonably useful and Bayesian methods provide superior ways of constructing more reliable HPD intervals especially when the sample size is small. The Bayesian approach and the asymptotic method yield similar results when the sample size is large.

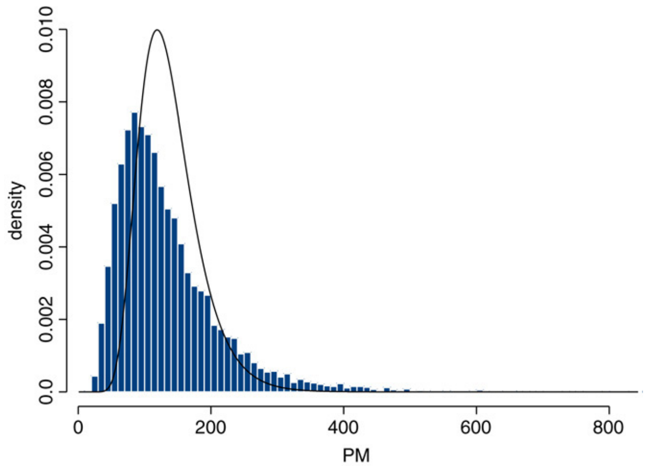

Fig. 11. Posterior distribution for *MTTF*: Jeffreys' prior ($n = 30$).

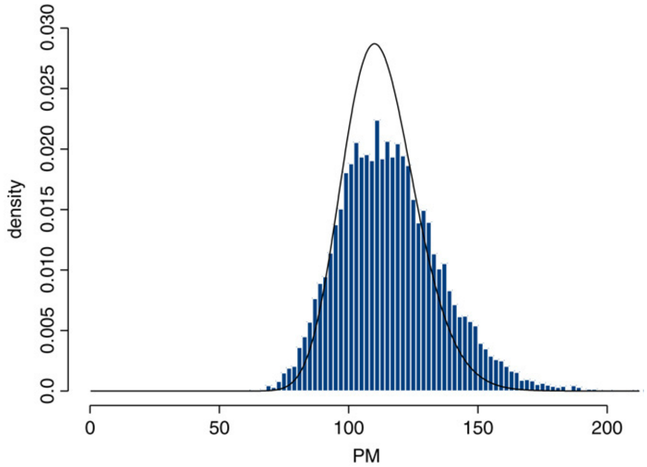

Fig. 12. Posterior distribution for *MTTF*: Jeffreys' prior ($n = 300$).

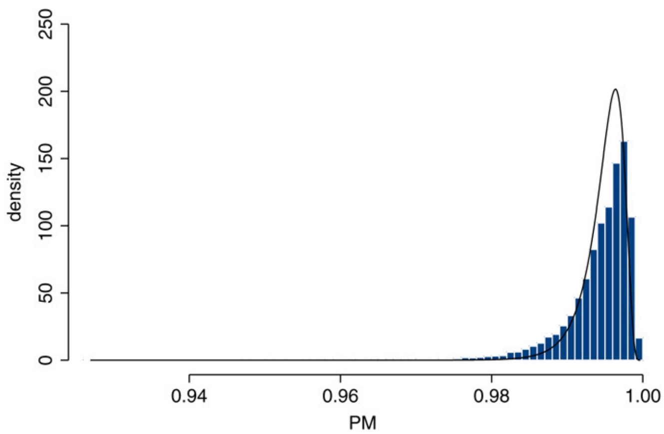

Fig. 13. Posterior distribution for $A(\infty)$: Jeffreys' prior ($n = 30$).

5. Conclusions

The Bayesian approach presented in this paper, using different and appropriate prior distributions, provides an alternative way of dealing with a repairable system with an unreliable service station. The proposed method gives reliable

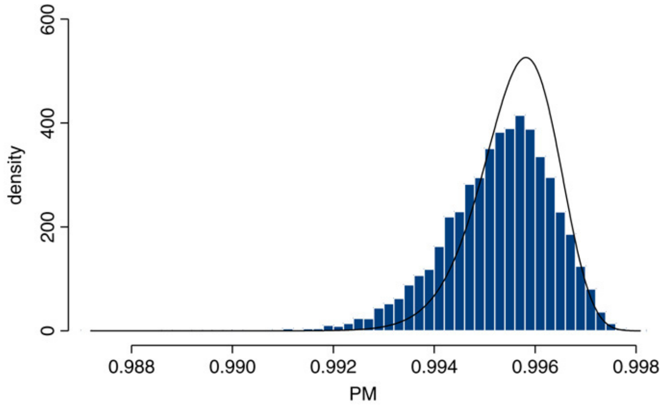

Fig. 14. Posterior distribution for $A(\infty)$: Jeffreys' prior ($n = 300$).

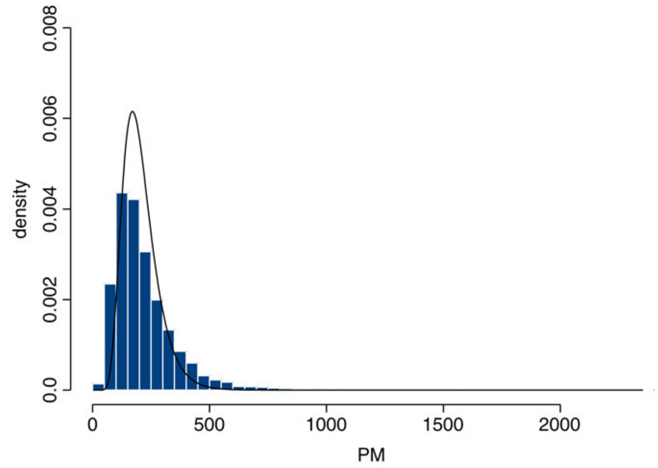

Fig. 15. Posterior distribution for *MTTF*: Beta distribution of the second kind prior ($n = 30$).

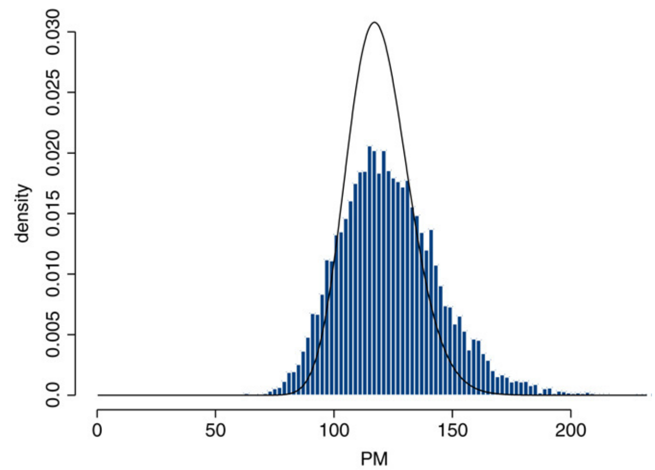

Fig. 16. Posterior distribution for *MTTF*: Beta distribution of the second kind prior ($n = 300$).

interval estimations for *MTTF* and $A(\infty)$ even when the sample size is small. According to the results of the simulation study, we found that λ has a large effect on the PM and β has a slight effect on the PM for *MTTF* and $A(\infty)$, respectively. Under the gamma prior and the beta distribution of the second kind prior with various hyperparemeters, we found that estimates of the PM for *MTTF* and $A(\infty)$ are close to the true value even though the sample size is small. Tables 5 and 6 clearly show the width of 95% HPD and PM varies under the different sample sizes when Jeffreys' prior is assumed. In Tables 9–11, comparing the

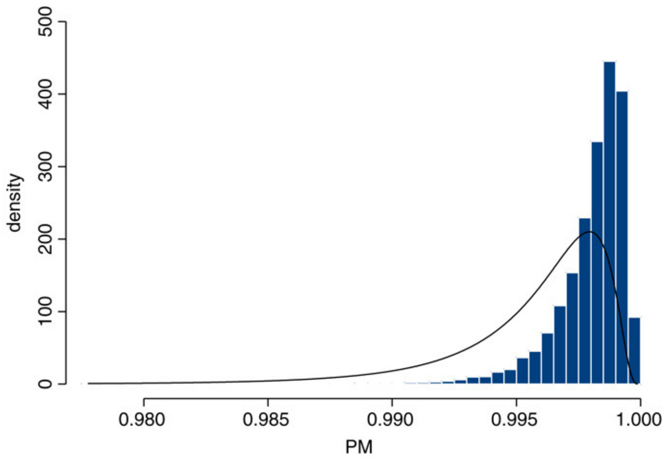

Fig. 17. Posterior distribution for $A(\infty)$: Beta distribution of the second kind prior ($n = 30$).

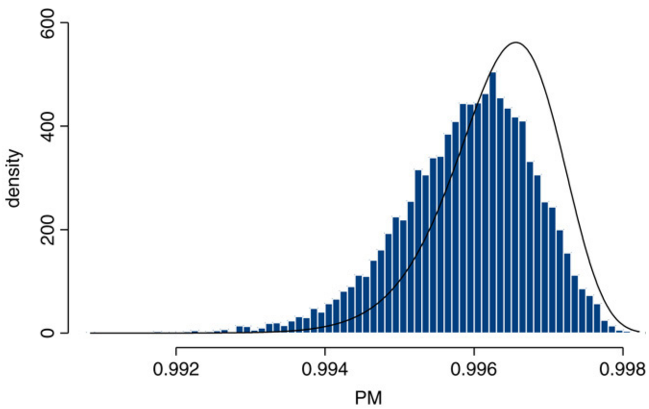

Fig. 18. Posterior distribution for $A(\infty)$: Beta distribution of the second kind prior ($n = 300$).

PM and the HPD intervals of the standard gamma prior with asymptotic estimates and the ACI. It implies that the spreads of posterior distributions and asymptotic distributions will decrease with increasing sample size. In the behavior of the figures, this also indicates that no matter what prior is chosen, the spreads of posterior distributions for *MTTF* and $A(\infty)$ will get smaller as the sample size gets larger. Furthermore, the computations involved are relatively easy. It is therefore fair to say that the Bayesian approach is quite useful and easy to implement in analyzing a repairable system with an unreliable service station when the prior is properly chosen.

References

[1] A. Birolini, The use of stochastic processes in molding reliability problems, in: Lecture Notes in Economics and Mathematical Systems, vol. 252, Springer Verlag, Berlin, 1985.
[2] R.D. Yearout, P. Reddy, D.L. Grosh, Standby redundancy in reliability—A review, IEEE Transactions on Reliability R-35 (1986) 285–292.
[3] J. Sztrik, Asymptotic analysis of a heterogeneous renewable complex system with random environments, Microelectronics and Reliability 32 (1992) 975–986.
[4] L.R. Goel, P. Shrivastava, Profit analysis of a two-unit redundant system with provision for test and correlated failures and repairs, Microelectronics and Reliability 31 (1991) 827–833.
[5] D.H. Shi, W. Li, Availability analysis of a two unit series system with shut-off rule and first-fail, first-repaired, Acta Mathematicae Applicatae Sinica 1 (1993) 88–91.
[6] A.T. de Almeida, F.M. Campello de Souza, Decision theory in maintenance strategy for a 2-unit redundant standby system, IEEE Transactions on Reliability 42 (3) (1993) 401–407.
[7] M. Gururajan, B. Srinivasan, A complex two-unit system with random breakdown of repair facility, Microelectronics and Reliability 35 (2) (1995) 299–302.
[8] D.H. Shi, L. Liu, Availability analysis of a two-unit series system with a priority shut-off rule, Naval Research Logistics 43 (1996) 1009–1024.
[9] S.P. Rajamanickam, B. Chandrasekar, Reliability measures for two-unit systems with a dependent structure for failure and repair times, Microelectronics and Reliability 37 (5) (1997) 829–833.
[10] R. Billinton, J. Pan, Optimal maintenance scheduling in a two identical component parallel redundant system, Reliability Engineering and System Safety 59 (1998) 309–316.
[11] V. Sridharan, P. Mohanavadivu, Some statistical characteristics of a repairable, standby, human & machine system, IEEE Transactions on Reliability 47 (4) (1998) 431–435.

[12] V.S.S. Yadavalli, M. Botha, A. Bekker, Asymptotic confidence limits for the steady-state availability of a two-unit parallel system with preparation time for the repair facility, Asia–Pacific Journal of Operational Research 19 (2002) 249–256.
[13] J.H. Seo, J.S. Jang, D.S. Bai, Lifetime and reliability estimation of repairable redundant system subject to periodic alternation, Reliability Engineering and System Safety 80 (2003) 197–204.
[14] M. Jie, Interval estimation of availability of a series system, IEEE Transactions on Reliability R-40 (5) (1991) 541–546.
[15] B.N. Masters, T.O. Lewis, W.J. Kolarik, A confidence interval availability for systems with Weibull operating time and lognormal repair time, Microelectronics and Reliability 32 (1992) 89–99.
[16] P. Chandrasekhar, R. Natarajan, V.S.S. Yadavalli, A study on a two unit standby system with Erlangian repair time, Asia-Pacific Journal of Operational Research 21 (3) (2004) 271–277.
[17] V.S.S. Yadavalli, A. Bekker, J. Pauw, Bayesian study of a two-component system with common-cause shock failures, Asia-Pacific Journal of Operational Research 22 (1) (2005) 105–119.
[18] S.K. Agarwal, S.L. Kalla, A generalized gamma distribution and its application in reliability, Communications in Statistics -Theory and Methods 25 (1) (1996) 201–210.
[19] W.Q. Meeker, L.A. Escobar, Statistical Methods for Reliability Data, John Wiley and Sons, New York, 1998.